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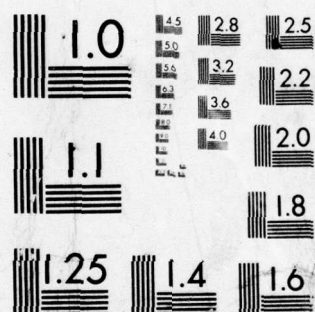
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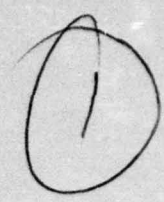
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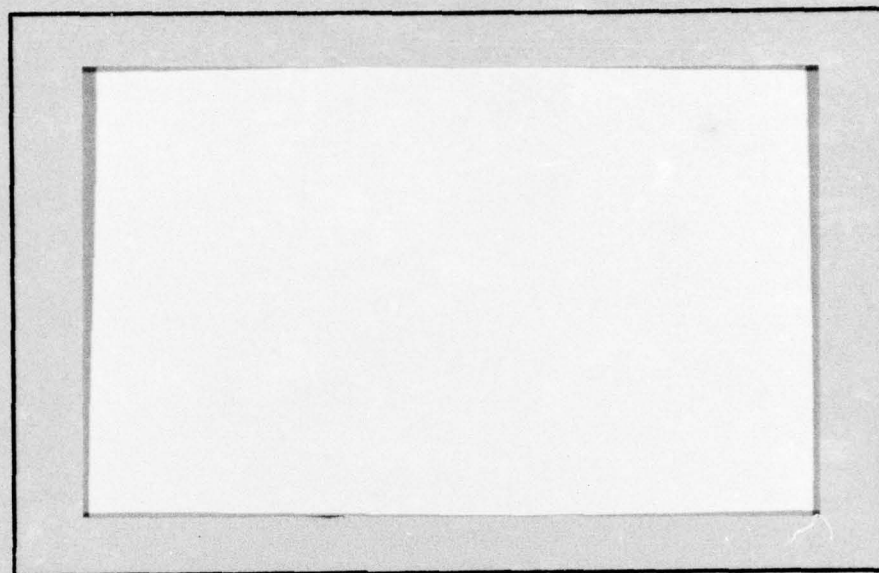


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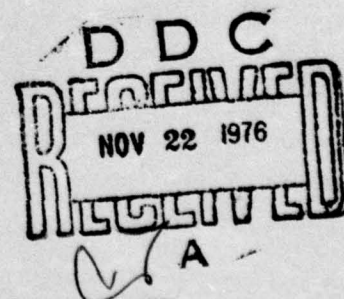
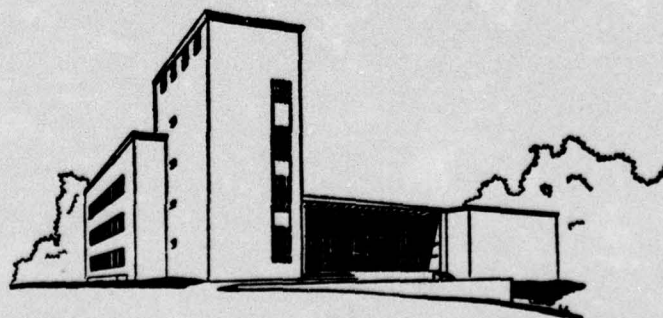


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A CONVERSE FOR DISJUNCTIVE CONSTRAINTS -

by

C. E. Blair  
Department of Business Administration  
University of Illinois, Urbana

R. G. Jeroslow  
GSIA, Carnegie-Mellon University

June 1976

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### Abstract

A necessary and sufficient condition is given, for the disjunctive constraints construction to provide all valid cuts for a system of logical constraints on linear inequalities.

### Key Words:

- 1) Integer programming
- 2) Disjunctive constraints
- 3) Polyhedral annexation
- 4) Cutting-planes

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# A CONVERSE FOR DISJUNCTIVE CONSTRAINTS

by C.E. Blair and R.G. Jeroslow

Recently programming problems have been considered which place logical restrictions on linear inequalities [1], [4], [8]. One format for these restrictions [1], [2], [3] involves several sets of linear constraints with the restriction that the feasible vectors satisfy all the constraints in at least one of the sets.

Formally, we are concerned with the set of  $x \in R^n$  satisfying a disjunction of inequality systems, i.e., the requirement that at least one of the following systems hold for  $x$ :

$$(S)_j \quad \begin{array}{l} A_j x \geq b_j \\ x \geq 0 \end{array} \quad (j \in H)$$

where  $b_j \in R^{m(j)}$ ,  $A_j \in R^{m(j) \times n}$ . Balas [1], [2] gave a method for obtaining inequalities

$$(1) \quad \theta x \geq \delta \quad [\theta \in R^n, \delta \in R]$$

such that every  $x$  satisfying at least one of the  $(S)_j$  satisfied (1):

$$(*) \quad \begin{array}{l} \text{If } h_j \in R^{m(j)}, h_j \geq 0 \text{ for } j \in H \text{ then (1) is a valid cut where} \\ \delta \leq \inf_{j \in H} h_j b_j \text{ and the} \\ i^{\text{th}} \text{ component of } \theta \text{ is at least the supremum (over } j \in H) \\ \text{of the } i^{\text{th}} \text{ component of } h_j A_j. \end{array}$$

In  $(S)_j$ , we allow the index set  $H$  to be finite or infinite so long as  $H \neq \emptyset$ . In  $(*)$ , it is required that the infima and the supremum mentioned exist.

In this note we investigate conditions under which the operation  $(*)$  gives all valid cuts for the systems  $S_j$ .

Let  $S \subseteq H$  be the subscripts of those systems in  $S_j$  which are consistent [i.e.,  $j \in S$  iff there is an  $x \geq 0$  with  $A_j x \geq b_j$ ]. Let  $I = H \setminus S$ . For  $j \in H$ ,  $\mathcal{C}(A_j) \subset \mathbb{R}^n$  is the cone generated by the rows of  $A_j$ .  $\mathcal{D}(A_j) = \{v \mid v \geq w \text{ for some } w \in \mathcal{C}(A_j)\}$  i.e.,  $\mathcal{D}(A_j)$  is the cone generated by the rows of  $A_j$  plus the rows of the identity matrix for the constraints  $x \geq 0$ . In what follows, the intersection over an empty index set is the whole space.

**Theorem 1:** The operation  $(*)$  gives all valid cuts for the disjunctive conditions  $(S)_j$  if

$$(\#) \quad \bigcap_{j \in S} \mathcal{D}(A_j) \subset \mathcal{D}(A_k)$$

holds for every  $k \in I$ . Conversely, if  $S$  is finite and  $(*)$  gives all valid cuts then  $(\#)$  holds for every  $k \in I$ .

**Proof:** Suppose  $(\#)$  holds and that  $\theta x \geq \delta$  is a valid cut. By the Farkas Lemma, for each  $j \in S$  there is  $h_j \geq 0$  such that  $h_j A_j \leq \theta$  and  $h_j b_j \geq \delta$ . By the Kuhn-Fourier Theorem, for each  $j \in I$  there is  $g_j \geq 0$  such that  $g_j A_j \leq \bar{0}$  and  $g_j b_j > 0$ . By  $(\#)$ ,  $\theta \in \mathcal{D}(A_j)$  for every  $j \in I$ , hence there is  $g'_j \geq 0$  such that  $g'_j A_j \leq \theta$ . So we may obtain (1) by the operation  $(*)$  by taking  $h_j$  as above for  $j \in S$  and  $h_j = g'_j + \lambda_j g_j$  for  $j \in I$  and  $\lambda_j$  sufficiently large.



On the other hand, if (\*) does not hold there is

$\theta \in \bigcap_{j \in S} \mathcal{A}(A_j) \setminus \bigcap_{j \in I} \mathcal{A}(A_j)$ . If  $S$  is finite there is some  $\delta$  such that the inequality  $\theta x \geq \delta$  is valid for the disjunctive system  $(S)_j$ . Since there is some  $j \in I$  such that  $h_j A_j$  is not componentwise  $\leq \theta$  for any  $h_j \geq 0$ , (\*) cannot yield  $\theta x \geq \delta$ . Q.E.D.

For  $j \in H$ ,  $\mathcal{A}(A_j)$  is a finitely generated cone. The polar of  $\mathcal{A}(A_j)$  is the negative of  $\{x | x \in \mathbb{R}^n, x \geq 0, A_j x \geq 0\}$ .

For finitely generated cones  $A \supseteq B \rightarrow A^P \subseteq B^P$ ,  $A^{PP} = A$ , and

$$(2) \quad \left( \bigcap_{i \in I} A_i \right)^P = \text{closure} \left\{ \sum_{i \in F} A_i^P \mid F \subseteq I, I \text{ finite} \right\}$$

for an arbitrary index set  $I$ . Note that, if  $I$  in (2) is finite,

(2) becomes

$$(2)' \quad \left( \bigcap_{i \in I} A_i \right)^P = \sum_{i \in I} A_i^P.$$

We have the following results, where the sum over an empty index set is  $\{0\}$ .

Corollary 1: If  $H$  is finite the operation (\*) gives all valid cuts for the systems  $(S)_j$  iff

$$(\#\#)' \quad \sum_{j \in S} \{x | x \geq 0, A_j x \geq 0\} \supseteq \{x | x \geq 0, A_k x \geq 0\}$$

for every  $k \in I$ .

Corollary 2: For any  $H$ , (\*) gives all valid cuts if

$$(\#\#) \quad \text{closure} \left( \bigcup_{\substack{F \subseteq S \\ F \text{ finite}}} \left\{ \sum_{j \in F} \{x | x \geq 0, A_j x \geq 0\} \right\} \right) \supseteq \{x | x \geq 0, A_k x \geq 0\}$$

for every  $k \in I$ .



We mention three applications of Corollary 2.

Corollary 3: ([6, Corollary 23] for  $|H|$  finite). If  $A_j = A$  is independent of  $j$  and  $S \neq \emptyset$  then (\*) gives all valid cuts.

Corollary 4: If  $S \neq \emptyset$  and for each  $k \in I$  there is  $j \in S$  such that

$$\{x | x \geq 0, A_j x \geq 0\} \supseteq \{x | x \geq 0, A_k x \geq 0\}$$

then (\*) gives all valid cuts.

Corollary 5: If each system  $A_j x \geq b_j$  has the form

$$\begin{array}{rcl} A'x & \geq & b'_j \\ x_1 & = & d_{j,1} \\ & \vdots & \\ & & \vdots \\ x_r & = & d_{j,r} \end{array}$$

then (\*) gives all valid cuts.

Proof:  $\{x | x \geq 0, A'x \geq 0, x_1 = 0, \dots, x_r = 0\}$  is independent of  $j$ , and Corollary 3 applies.

Q.E.D.

Corollary 5 shows that the disjunctive constraint systems corresponding to an integer program provide all the valid cutting-planes, hence the same is also true of the equivalent polyhedral annexation construction (see also [4], [5] for a proof of Corollary 5 for a bounded integer program).

When  $S$  is infinite the operation  $(*)$  may yield all valid inequalities even though  $(\#)$  does not hold.

Example: Let  $n = 2$ ,  $H = \{0, 1, 2, \dots\}$ . Consider the systems

$$\begin{array}{ll} x_1 \geq 2 & -x_2 \geq -1 \\ (S)_0 \text{ is } -x_1 \geq -1 & \text{for } i \geq 1, (S)_i \text{ is } x_1, x_2 \geq 0 \\ x_1, x_2 \geq 0 & \end{array}$$

Clearly every  $(x_1, x_2) \in \mathbb{R}^2$  with  $x_1, x_2 \geq 0$  satisfies at least one of the systems  $(S)_i$ , so the valid cuts are all of the form  $\alpha x_1 + \beta x_2 \geq \sigma$  for all  $\alpha, \beta \geq 0$  and  $\sigma \leq 0$ . These can clearly be obtained by  $(*)$ .

However  $S = \{1, 2, \dots\}$ ,  $I = \{0\}$ , and

$$\theta = (0, -1) \in \bigcap_{j \in S} (A_j) \setminus (A_0)$$

so  $(\#)$  fails.

Finally, we give a necessary and sufficient condition for the case when  $S$  is infinite. This condition appears to be harder to use than the sufficient condition  $(\#\#)$ . For  $i \in S$ , let  $E_i = \{x \geq 0 \mid A_i x \geq b_i\}$ .

Theorem 6: The operation  $(*)$  gives all valid inequalities for  $(S)_j$  iff

$$(\#\#\#) \quad \text{clconv}\left(\{0\} \cup \left(\bigcup_{i \in S} E_i\right)\right) \supseteq \{x \geq 0 \mid A_j x \geq 0\}$$

for every  $j \in I$ .

Proof. If  $\theta x \geq \delta$  is a valid inequality for the systems  $(S)_j$  then every  $x \in \bigcup_{i \in S} E_i$  must satisfy  $\theta x \geq \delta$ . By the Farkas lemma, for each  $j \in S$  there is  $g_j \geq 0$  such that  $g_j A_j \leq \theta$  and  $g_j b_j \geq \delta$ . For every  $x \in \text{clconv}\left(\{0\} \cup \left(\bigcup_{i \in S} E_i\right)\right)$

we have  $\theta x \geq \min(0, \delta)$ . If (###) holds, then the conditions  $x \geq 0$ ,  $A_j x \geq 0$  imply that  $\theta x \geq \min(0, \delta)$  whenever  $j \in I$ . By the Farkas lemma there is, for each  $j \in I$ ,  $h_j \geq 0$  such that  $h_j A_j \leq \theta$ . As in the proof of Theorem 1, the inconsistency of  $A_j x \geq b_j$ ,  $x \geq 0$  is used to obtain  $\theta x \geq \delta$  by (\*).

Conversely, if (###) fails there is  $j \in I$  and  $z \in \{x \geq 0 \mid A_j x \geq 0\} \setminus \text{clconv}\left(\{0\} \cup \left(\bigcup_{i \in S} E_i\right)\right)$ . By the separating hyperplane theorem there is  $\theta, \delta$  such that  $\theta z < \delta$  and  $\theta x \geq \delta$  for  $x \in \text{clconv}\left(\{0\} \cup \left(\bigcup_{i \in S} E_i\right)\right)$ .  $\theta x \geq \delta$  is valid for the systems  $(S)_j$  and also  $0 = \theta \cdot 0 \geq \delta$ .

If  $\theta x \geq \delta$  were obtained by (\*), for some  $h_j \geq 0$  we have  $h_j A_j \leq \theta$ . Hence, as  $z \geq 0$  and  $A_j z \geq 0$ , also  $\theta z \geq h_j A_j z \geq 0 \geq \delta$ . This contradicts  $\theta z < \delta$ .

Q.E.D.

May 20, 1976



#### REFERENCES

1. E. Balas "On the Use of Intersection Cuts and Outer Polars in Branch-and-Bound," MPS conference talk, Stanford, August, 1973.
2. E. Balas, "Intersection Cuts from Disjunctive Constraints," MSRR no. 330, Carnegie-Mellon University, February 1974.
3. E. Balas, "Disjunctive Programming: Cutting-Planes from Logical Conditions," talk at SIGMAP-UW Conference, Madison, 1974. Published in [7].
4. F. Glover, "Polyhedral Annexation in Mixed Integer Programming," MSRS 73-9, University of Colorado, August 1973.
5. F. Glover, "Polyhedral Annexation in Mixed Integer and Combinatorial Programming," Mathematical Programming 9 (1975), pp. 161-188.
6. R. Jeroslow, "Cutting-Planes for Relaxations of Integer Programs," MSRR no. 347, Carnegie-Mellon University, July 1974.
7. O.L. Mangasarian, R.R. Meyer, and S.M. Robinson, Nonlinear Programming 2, Academic Press, 1975. 360<sup>+</sup> pp.
8. G. Owen, "Cutting-Plane for Programs with Disjunctive Constraints," Journ. Opt. Th. and Appl. 11 (1973).



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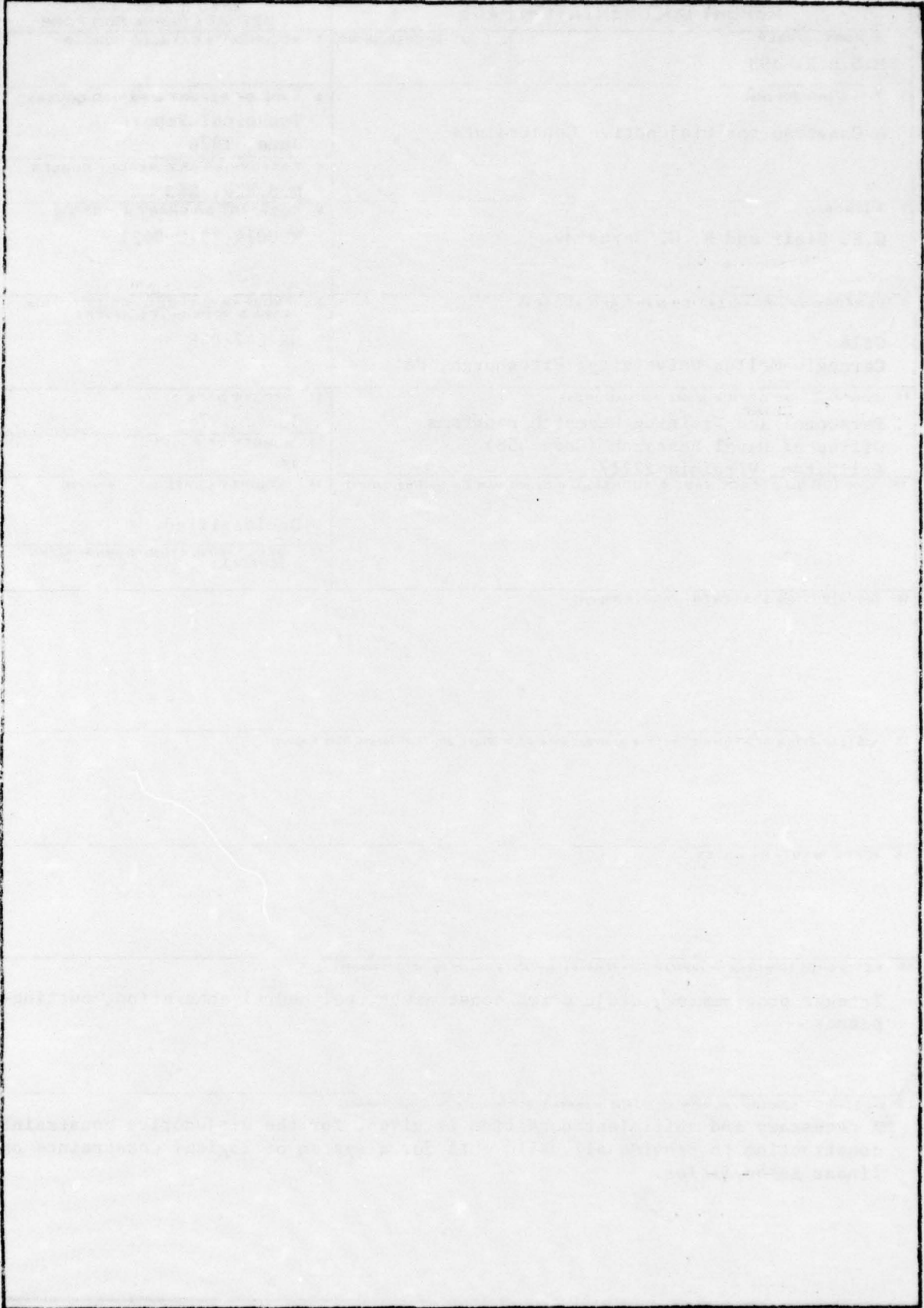
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